GENERAL FORMULA FOR COLLISION IMPULSE OF TWO MATERIAL OBJECTS IN TRANSLATIONAL MOTION

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Dedicated to Homer the dog.

ABSTRACT. This article is a part of documentation of the **GPX** library. It describes methods, algorithms and derivations of formulas used in **GPX** .

1. General formula for collision impulse

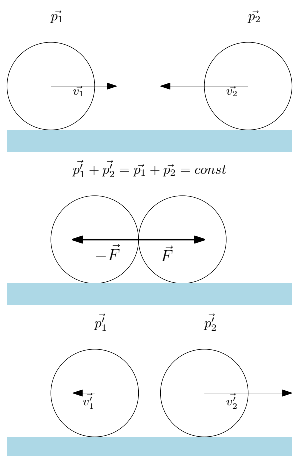

Figure 1. Colliding bodies

1.1. **Two material objects in translational motion.** Whenever two material objects collide they exert forces on each other for a period of a collision. Newton's third law states that this forces are always exact in magnitude, but opposite in direction. In real time simulations we often expect collision time to be very short, that is collision is performed in a time shorter than a single step of simulation. We are not interested in exact distribution of collision force over that time. Considering the above and denoting collision time as dt, momentum of the first body as $\vec{p_1}$ and momentum of the second body as $\vec{p_2}$, we can define equations for post-collision momentums:

$$\begin{cases} \vec{p_1'} = \vec{p_1} - \vec{F}dt \\ \vec{p_2'} = \vec{p_2} + \vec{F}dt. \end{cases}$$

Now we would like to find the impulse $\vec{F}dt$. Assuming that the direction of impulse $\vec{F}dt$ is known, let's focus on its magnitude. There are two edge cases we could find Fdt for. These are:

(a) perfectly elastic collisions,
(b) perfectly inelastic ones.

In terms of physics, perfectly elastic collisions are those in which kinetic energy of colliding bodies is conserved[1]. Perfectly inelastic collisions are those in which kinetic energy loss is maximal. That happens when velocities are equalized after the collision, what is intuitively perceived as objects being sticked together. Note that no matter how we choose the magnitude, the total momentum of the system is preserved[2].

Perfectly elastic collision between two bodies can be expressed as below. For simplicity we will consider one dimensional case. Conservation of the kinetic energy in the system is the entry point.

$$\begin{aligned} & E_{k1}' + E_{k2}' = E_{k1} + E_{k2} \\ & \frac{{p_1'}^2}{2m_1} + \frac{{p_2'}^2}{2m_2} = \frac{p_1^2}{2m_1} + \frac{(-p_2)^2}{2m_2} \\ & \frac{(p_1 - Fdt)^2}{2m_1} + \frac{(-p_2 + Fdt)^2}{2m_2} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \end{aligned}$$

We have assumed that $\vec{p_2}$ vector is pointing to direction opposite to $\vec{p_1}$ and put minus sign near its scalar value. After some transformations we may receive result

[1]There is distinction between something being *preserved* and *conserved*. By calling something conserved we mean it does not change at the output of some process, yet we don't care what happens in process duration. On the other hand, something is preserved when its state is conserved all the time. The building may have been damaged during the war, then restored to its previous state and we would say that it is conserved. However, if building was untouched by the war, we may say it is preserved.

[2]See 1.

in the following form:

$$F dt \left(\frac{1}{m_1} + \frac{1}{m_2} \right) = \frac{2p_1}{m_1} + \frac{2p_2}{m_2}. \tag{1.1}$$

Now let's take a look at perfectly inelastic collision. It can be expressed straightly as equalization of post-collision velocities.

$$v_1' = v_2'$$
$$\frac{p_1 - F dt}{m_1} = \frac{-p_2 + F dt}{m_2}$$

Again, the result can be transformed into the form:

$$F dt \left(\frac{1}{m_1} + \frac{1}{m_2} \right) = \frac{p_1}{m_1} + \frac{p_2}{m_2}. \tag{1.2}$$

When we compare equations (1.1) and (1.2), we will point out that they differ only by coefficients near momentums on the right-hand side, which in case of perfectly elastic collision is equal 2. This coefficient may be considered as a measure of elasticity and it can parametrize equation:

$$F dt \left(\frac{1}{m_1} + \frac{1}{m_2} \right) = \frac{elasticity_1\ p_1}{m_1} + \frac{elasticity_2\ p_2}{m_2}. \tag{1.3}$$

Without referring to particular body, may $elasticity_1 = elasticity_2 = elasticity$. Edge cases for perfectly elastic and perfectly inelastic collisions are defined respectively by $elasticity \in \{2, 1\}$. Intermediate collision types, that are neither perfectly elastic nor perfectly inelastic are defined by $elasticity \in (1, 2)$. Values other than above make less sense in classical mechanics. If $elasticity \in (2, \infty)$, it would mean that system gained some energy during collision. The effect of $elasticity \in (0, 1)$ would be objects tunneling each other. In a case of $elasticity = 0$ the system would behave like there had been no collision. Negative values would reverse the force action. Whenever $elasticity_1 \neq elasticity_2$ some kind of intermediate collision type is performed.

Let's generalize equation (1.3) one step further.

$$F dt\, (a_1 + a_2) = b_1 + b_2 \tag{1.4}$$

Symbolically:

$$A\, F dt = B. \tag{1.5}$$

Denoting that $A = \sum a_i$ and $B = \sum b_i$, A has a dimension of $\left[{}^{1}/_{mass}\right]$ and B has a dimension of $[velocity]$. Sometimes we will refer to the equation (1.5) as *form* $A\, F dt = B$. We will find out that such form is very convenient for purpose of collision engine. Obviously, impulse magnitude $F dt = \frac{B}{A}$.

1.1.1. *More axes.* We derived the formula for one dimensional collision. However, it can be applied to any direction of the basis. Thus (1.5) can be presented in vectorized form:

$$A \vec{F} dt = \vec{B}.$$

By appropriate selection of basis, separate collision impulses can be defined for bounciness and friction.

Collisions of multi-body systems can be handled by system of such equations.

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